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NECESSARY AND SUFFICIENT CONDITION FOR EXISTENCE A k- DIMENSION A- CONINVARIANT SUBSPACE OF C^n

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Abstract. We know λ is a coneigenvalue of a matrix $A \in C^{n\times n}$ if there exists a nonzero vector $x \in C^n$ such that $A\bar{x} = \lambda x$. Coneigenvalues defined in this way may exist if and only if $\bar{A}A$ has real nonnegative (ordinary) eigenvalues. If A does have coneigenvalues, there are always infinitely many of them. We attempt a number of new facts on coninvariant subspaces and block triangular matrices consimilarity with them. In this paper, we show that each square matrix is consimilar with block triangular matrix if and only if has a nontrivial A-coninvariant subspace.

Keywords: consimilarity, coneigenvalue, coneigenvector, coninvariant subspace.

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1 Introduction

As we know, similarity has particular importance in Matrix Analysis, because some of the characteristics are preserved as eigenvalues. Given the definition of coneigenvalue in Horn & Johnson (2013), we are following the transformation that preserves all of coneigenvalues. These transformations have the same consimilarity. We know matrices $A, B \in C^{n\times n}$ are said to be consimilar if $A = SB\bar{S}^{-1}$ for a nonsingular matrix $S \in C^{n\times n}$, where, as usual, $\bar{S}$ is the component-wise conjugate of S. Unitary congruence is an particularly important case of consimilarity obtained when $S = U$ is a unitary matrix: $A = UBU^T$. There exists an extensive literature on consimilarity and unitary congruence, which provides a rather complete theory for these matrix relations. Consimilarity has a very long history, going back to Segre (1890) and perhaps earlier. In this paper we have considering eigenvalue definition in the sense of Fassbender & Ikramov (2007), Ikramov (2007), Ikramov (2010). The following theorems are formulated and proved in Hong & Horn (1986), Horn & Johnson (2013).

Theorem 1. *Let $A \in M_n$ be given. Then A is similar to a block matrix of the form*

$$\begin{pmatrix} \Lambda & C \\ 0 & D \end{pmatrix}, \quad \Lambda = diag(\lambda_1, ..., \lambda_k), D \in M_{n-k}(C), 1 \leq k < n$$

if and only if there are k linearly independent vectors in C^n, each of which is an eigenvector of A. If $x^{(1)}, \dots, x^{(n)}$ are linearly independent coneigenvectors of A and if $S = [x^{(1)} \dots x^{(n)}]$, then $S^{-1}A\bar{S}$ is a diagonal matrix.

Proof. (Horn & Johnson, 2013).
Also

Theorem 2. *Suppose that $n \geq 2$, $A \in M_n(C)$ is similar with block triangular form $\begin{pmatrix} B_{k\times k} & C \\ 0 & D \end{pmatrix}$, if and only if has a nontrivial $k-$dimensional A-invariant subspace of C. Also if $L \subseteq \boldsymbol{C}^n$ is nonzero A-invariant subspace, then, there exists a vector in L such that, it is eigenvector of A.*

Proof. (Horn & Johnson, 2013).

In the next section, we proof consimilarty state of these theorems.

2 Main results

In reference Ghasemi Kamalvand (2013) had been shown that if $A \in M_n(\mathbf{C})$, Then A is condiagonalizable if and only if there is a set of n linearly independent vectors, each of which is a coneigenvector of A.

In this section, we show more general case than in Ghasemi Kamalvand (2013). Latter theorem affirmation this topic.

Theorem 3. *Let $A \in M_n(C)$ be given. Then A is consimilar to a block matrix of the form*

$$\begin{pmatrix} \Lambda & C \\ 0 & D \end{pmatrix}, \quad \Lambda = diag(\mu_1, ..., \mu_k), D \in M_{n-k}(C), 1 \leq k < n \tag{1}$$

if and only if there are k linearly independent vectors in C^n, each of which is an coneigenvector of A. If $x^{(1)}, \ldots, x^{(n)}$ are linearly independent coneigenvectors of A and if $S = [x^{(1)} \ ... \ x^{(n)}]$, then $S^{-1}A\bar{S}$ is a diagonal matrix.

Proof. Suppose that $k < n$, the k-vectors $x^{(1)}, ..., x^{(k)}$ are linearly independent, and let

$$S_1 = [x^{(1)} ... \ x^{(k)}], S_2 = [x^{(k+1)} ... \ x^{(n)}]$$

such that $S = [S_1 \ S_2]$ is nonsingular. Since $x^{(1)}, x^{(2)}, ..., x^{(k)}$ are linearly independent, then we can expand this set to a basis of C^n. The first show that $S^{-1}A\overline{S}$ has this form $\begin{pmatrix} \Lambda & C \\ 0 & D \end{pmatrix}$.

$$A\overline{S} = [A\overline{x^{(1)}} \ldots A\overline{x^{(k)}} \quad A\bar{S}_2] = [\mu_1 x^{(1)} \mu_2 x^{(2)} \ldots \mu_k x^{(k)} \quad A\bar{S}_2]$$

then

$$S^{-1}A\bar{S} = [\mu_1 e_1 \ \mu_2 e_2 \quad \ldots \quad \mu_k e_k \quad S^{-1}A\bar{S}_2] = \begin{pmatrix} I_k\Lambda & C \\ 0 & S^{-1}A\bar{S}_2 \end{pmatrix} = \begin{pmatrix} \Lambda & C \\ 0 & D \end{pmatrix}$$

.

Conversely, suppose that A is consimilar with $\begin{pmatrix} \Lambda & C \\ 0 & D \end{pmatrix}$, then exist $S \in M_n(C)$ is nonsingular such that $S^{-1}A\bar{S} = \begin{pmatrix} \Lambda & C \\ 0 & D \end{pmatrix}$.

Let $S = [S_1 \ S_2]$ as $S_1 \in M_{n,k}, S_2 \in M_{n,n-k}$.

$$A\bar{S} = S\begin{pmatrix} \Lambda & C \\ 0 & D \end{pmatrix} = [S_1\Lambda \quad S_1C + S_2D] \Rightarrow A\overline{S_1} = S_1\Lambda,$$

so each column of S_1, is coneigenvector of A. Now if $x^{(1)}, ..., x^{(k)}$, are the first k columns of nonsingular matrix S and $k < n$ then these columns are linear independent.

If $k = n$ then $S^{-1}A\bar{S}$ is diagonal matrix and Λ is n-dimensional.(this is identical with theorem (4.1) of Ghasemi Kamalvand (2013)). ☐

Theorem 4. *Suppose that $n \geq 2$, $A \in M_n(C)$ is consimilar with block triangular form $\begin{pmatrix} B_{k\times k} & C \\ 0 & D \end{pmatrix}$, if and only if C^n has a nontrivial $k-$dimensional A-coninvariant subspace.*

Proof. Suppose that $L \subseteq \mathbf{C}^n$ is a nontrivial $k-$dimensional A-coninvariant subspace, then it has a $k-$dimensional basis $\{S_1, S_2, ..., S_k\}$. This set can be expanded to a linearly independent set that is a basis on $\mathbf{C}^n$. Let $S = [S_1 \ ... \ S_k \ S_{k+1} \ ... \ S_n]$, then

$$S_1 = [S_1 \quad S_2 \ldots \quad S_k] \in M_{n,k} \quad , \quad S_2 = [S_{k+1} \quad S_{k+2} \ldots \quad S_n] \in M_{n,k}.$$

S is nonsingular matrix, because of its columns are linear independent.

$$S^{-1}A\bar{S} = S^{-1}A[\overline{S_1} \quad \overline{S_2}] = S^{-1}[A\overline{S_1} \quad A\overline{S_2}].$$

Since L is an A-coninvariant subspace, then for every $S_i \in L, 1 \le i \le k$ have $A\overline{S_i} \in L$ this means that $A\overline{S_i}$ is a linear combination of elements of basis L. Then considering

$$A\overline{S_i} = b_{1i}S_1 + b_{2i}S_2 + ... + b_{ki}S_k \Rightarrow A\overline{S_i} = S_iB, \quad B \in M_k$$

we have

$$S^{-1}A\bar{S} = S^{-1}[A\overline{S_1} \quad A\overline{S_2}] = S^{-1}[S_1B \quad A\overline{S_2}] = [S^{-1}S_1B \quad S^{-1}A\overline{S_2}]. \tag{2}$$

Therefore

$$S^{-1}S = [S_1 \;\; S_2] = [S^{-1}S_1 \;\; S^{-1}S_2] = I = \begin{pmatrix} I_k & 0 \\ 0 & I_{n-k} \end{pmatrix} \Rightarrow S^{-1}S = \begin{pmatrix} I_k \\ 0 \end{pmatrix} \tag{3}$$

With substitute (3) to (2), we have

$$S^{-1}A\bar{S} = [\begin{pmatrix} I_k \\ 0 \end{pmatrix} B, \quad S^{-1}AS_2] = \begin{pmatrix} B_{k\times k} & C \\ 0 & D \end{pmatrix}.$$

Conversely, suppose that exists nonsingular matrix $S \in M_n(C)$ such that

$$S^{-1}A\bar{S} = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix} \qquad B \in M_k, 1 \le k \le n.$$

We show that $\mathbf{C}^n$ has a $k-$dimensional A-coninvariant subspace.
Let $S = [S_1 \quad S_2]$, $S_1 \in M_{n,k}$ such that $S_1 = [S_1^{(1)} \ \ldots S_1^{(k)}]$ and $S_1^{(i)}, i = 1, 2..., k$ are columns of the matrix S_1 and assume that

$$L := Span\{S_1^{(1)}, S_1^{(2)}, ..., S_1^{(k)}\}.$$

Then we have

$$A\overline{S_1} = A\bar{S}\begin{pmatrix} I_k \\ 0 \end{pmatrix} = S\begin{pmatrix} B & C \\ 0 & D \end{pmatrix}\begin{pmatrix} I_k \\ 0 \end{pmatrix} = [S_1 \;\; S_2]\begin{pmatrix} B & C \\ 0 & D \end{pmatrix}\begin{pmatrix} I_k \\ 0 \end{pmatrix} = S_1B$$

$$\Rightarrow A\overline{S_1} = S_1B = b_{1i}S_1^{(1)} + b_{2i}S_1^{(2)} + ... + b_{ki}S_1^{(k)} \Rightarrow A\overline{S_1} \in L$$

i.e. L is a A-coninvariant subspace of $\mathbf{C}^n$. □

3 Conclusion

The important conclusion of this paper is existence a $k-$ dimensional A-coninvariant subspace of C^n.

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